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Solving general gauge theories on inner product spaces.

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Abstract

By means of a generalized quartet mechanism we show in a model independent way that a BRST quantization on an inner product space leads to physical states of the form

$$|ph\rangle = e^{[Q,\psi]}|ph\rangle_0$$

where Q is the nilpotent BRST operator, ψ a hermitian fermionic gauge fixing operator, and $|ph\rangle_0$ BRST invariant states determined by a *hermitian* set of BRST doublets in involution. $|ph\rangle_0$ does not belong to an inner product space although $|ph\rangle$ does. Since the BRST quartets are split into two sets of hermitian BRST doublets there are two choices for $|ph\rangle_0$ and the corresponding ψ. When applied to general, both irreducible and reducible, gauge theories of arbitrary rank within the BFV formulation we find that $|ph\rangle_0$ are trivial BRST invariant states which only depend on the matter variables for one set of solutions, and for the other set $|ph\rangle_0$ are solutions of a Dirac quantization. This generalizes previous Lie group solutions obtained by means of a bigrading.

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1 Introduction.

In ref.[1] simple expressions for the solutions of a BFV-BRST quantization on inner product spaces was obtained for arbitrary irreducible Lie group gauge theories with finite number of degrees of freedom. More precisely it was shown that provided one makes use of dynamical Lagrange multipliers and antighosts the physical states $|ph\rangle$, satisfying $Q|ph\rangle = 0$, may be written as

$$|ph\rangle = e^{[Q,\psi]}|ph\rangle_0 \tag{1.1}$$

where ψ is a specific hermitian fermionic gauge fixing operator with ghost number minus one, and where $|ph\rangle_0$ is a trivial BRST invariant state which only depends on the matter variables. These are formal solutions and one has to associate certain quantum prescriptions of the involved operators in order for these formal solutions to be true nontrivial solutions. The basic general quantization rule is that the unphysical degrees of freedom represented by ghosts and antighosts, Lagrange multipliers and gauge degrees of freedom are to be quantized in an opposite manner, *i.e.* one with positive and the other with indefinite metric states so that they together form states built of half positive and half indefinite metric state spaces [2]. (Further properties of these solutions are given in [3] and [4].)

In [5] it was shown that $|ph\rangle$ may also be related to solutions of a Dirac quantization. This relation was also shown to be of the form (1.1) but where the ghost fixed states $|ph\rangle_0$ are solutions of a Dirac quantization, and where ψ has to be chosen differently.

In this paper we give a general setting for solutions of the form (1.1) and prove that the results of [1, 5] may be generalized to arbitrary, both irreducible and reducible gauge theories.

Throughout the paper we make use of supercommutators defined by

$$[A, B] = AB - BA(-1)^{\varepsilon_A \varepsilon_B} \tag{1.2}$$

where ε_A and ε_B are the Grassmann parities of the operators A and B respectively. ($\varepsilon_A = 0$ for even A and $\varepsilon_A = 1$ for odd A.)

The paper is organized as follows: In section 2 we review some general properties of BRST quantization on inner product spaces. In section 3 we prove formula (1.1) in a general setting and discuss some of its properties, and in section 4 we illustrate these properties in a simple case corresponding to *e.g.* abelian gauge theories. In section 5 we start to look for more explicit realizations. We consider first the minimal sector of general irreducible gauge theories of arbitrary rank within the BFV scheme. In section 6 we treat the nonminimal sector with dynamical Lagrange multipliers and antighosts. In section 7 we apply the results of section 3 and give some properties and interpretations. In section 8 we extend the previous results to general reducible gauge theories. Finally we conclude the paper in section 9.

2 BRST quantization on inner product spaces

Consider a gauge theory with a conserved nilpotent BRST charge operator Q. Let, furthermore, the associated state space V be a nondegenerate inner product space. The

states in V may then be subdivided into singlets and doublets under Q as follows ([6, 7]):

$$\begin{array}{ll} 1) & |s\rangle \in V \text{ is a singlet if } Q|s\rangle = 0, \quad |s\rangle \neq Q|u\rangle \text{ any } |u\rangle \in V \\ 2) & |d\rangle,\, |p\rangle \in V \text{ is a doublet if } |d\rangle = Q|p\rangle \neq 0 \end{array} \tag{2.3}$$

This subdivision is not unique since it is invariant under

$$\begin{aligned} |s\rangle &\to |s\rangle + |d'\rangle \\ |p\rangle &\to |p\rangle + |s'\rangle + |d''\rangle \end{aligned} \tag{2.4}$$

We may therefore impose further conditions. First we may choose $|d'\rangle$ and $|s'\rangle$ in (2.4) so that

$$\langle s|p\rangle = 0, \quad \forall\, |s\rangle,\ |p\rangle \tag{2.5}$$

is always valid [7]. In this case V is divided into a direct sum of singlet and doublet states:

$$V = V_S \oplus V_D \tag{2.6}$$

The nondegeneracy of the inner product of V forbid the existence of $|d\rangle$'s different from zero such that $\langle d|p\rangle = 0$ for all $|p\rangle$ since this condition is equivalent to $\langle d|u\rangle = 0$ for all $|u\rangle \in V$. Eq.(2.6) implies therefore that V_S is a representation of the BRST cohomology $KerQ/ImQ$ ($|s\rangle,\ |d\rangle \in KerQ,\ \ |d\rangle \in ImQ$). We may furthermore choose a $|d''\rangle$ to every $|p\rangle$ in (2.4) so that [7]:

$$\langle p|p'\rangle = 0 \tag{2.7}$$

for all $|p\rangle$-states. In the following we always require (2.5) and often (2.7) as well.

One way to determine V_S is through the Hodge decomposition implied by the coBRST charge [7, 8, 9, 10]. This construction requires the existence of an even, hermitian metric operator η satisfying

$$\langle u|\eta|u\rangle \geq 0,\ \ \forall |u\rangle \in V, \quad \eta^2 = \mathbf{1} \tag{2.8}$$

This means that V is a bilinear form on a Hilbert space which is a natural restriction. The coBRST charge is then defined in terms of η through

$$^*Q \equiv \eta Q \eta \tag{2.9}$$

This definition implies that $^*Q$ is nilpotent. One may now show that all $|p\rangle$'s may be chosen to have the form

$$|p\rangle = {}^*Q|u\rangle \tag{2.10}$$

which automatically satisfies (2.7) while (2.5) requires

$$^*Q|s\rangle = 0 \tag{2.11}$$

Since one may show that

$$\triangle|s\rangle = 0 \ \Leftrightarrow\ Q|s\rangle = {}^*Q|s\rangle = 0 \tag{2.12}$$

where $\triangle \equiv [Q, {}^*Q]_+$, V_S is also the space of BRST harmonic states. Eq.(2.6) is then the Hodge decomposition ($|u\rangle = |s\rangle + Q|u'\rangle + {}^*Q|u''\rangle$ any $|u\rangle \in V$).

There is no unique relation between the metric operator η and the coBRST charge ${}^*Q$. We may *e.g.* factorized η as follows

$$\eta = \eta_S\eta_D = \eta_D\eta_S \tag{2.13}$$

where

$$[Q, \eta_S] = 0, \quad [Q, \eta_D] \neq 0 \tag{2.14}$$

This implies

$${}^*Q = \eta_D Q \eta_D \tag{2.15}$$

which in turn implies that η_D determines all the properties within the coBRST approach. Notice that η_D must be nontrivial which means that the unphysical degrees of freedom must contain indefinite metric states. In fact, the unphysical degrees of freedom must be quantized with half positive and half indefinite metric states since the nondegenerate doublet space V_D is divided into two equally large subspaces of zero norm states, *i.e.* ImQ and Im^*Q. Notice that the singlet states $|s\rangle$ only have positive norms if $\eta_S = \mathbf{1}$, which thus is a condition one has to impose on physical theories. It is essentially equivalent to the completeness condition of Spiegelglas [8]: All zero norm states in $KerQ$ are in ImQ. (However, this condition also allow for $\eta_S = -\mathbf{1}$.)

We now turn to another way to determine V_S in (2.6) which has a less invariant form but which we expect to be related to the above approach. The starting point is the following argument: If we assume that the BRST doublets $|d\rangle$ and $|p\rangle$ may be represented as follows (this depends on the basis of V)

$$|p_i\rangle = C_i^\dagger|u\rangle, \quad |d_i\rangle = QC_i^\dagger|u\rangle \text{ any } |u\rangle \in V \tag{2.16}$$

then condition (2.5) requires

$$C_i|s\rangle = 0 \tag{2.17}$$

which in turn implies

$$B_i|s\rangle = 0, \quad B_i \equiv [Q, C_i] \tag{2.18}$$

The operator doublets, $D_r \equiv (C_i, B_i)$, must then satisfy the consistency conditions

$$[D_r, D_s] = K_{rs}{}^t D_t \tag{2.19}$$

which implies

$$[D_r^\dagger, D_s^\dagger] = D_t^\dagger K^{\dagger\,t}_{sr} \tag{2.20}$$

(That auxiliary conditions of the form (2.17)-(2.18) may always be imposed was demonstrated in [11, 12].) The nondegeneracy of the inner product of V_D requires now

$$[D_r, D_s^\dagger] \text{ is an invertible matrix operator} \tag{2.21}$$

even between singlet states. From (2.19) and (2.20) this requires that the set $\{D_r\}$ is linearly independent of $\{D_r^\dagger\}$ and that they together constitute a set of (generalized) BRST quartets [6, 7, 12]. One may notice that if singlet states are assumed to be of the form

$$|s_i\rangle = A_i^\dagger|0\rangle \tag{2.22}$$

where $|0\rangle$ is a singlet vacuum state then the singlet operators must satisfy

$$\begin{aligned} [D_r, A_i^\dagger] &= a_{ir}{}^s D_s \\ [A_i, D_r^\dagger] &= D_s^\dagger a_{ir}^{\dagger\,s} \end{aligned} \tag{2.23}$$

for consistency.

The property (2.7) of the $|p\rangle$-states requires at least that the commutator $[C_i, C_j^\dagger]$ vanish between singlet states. This together with the property

$$[B_i, B_j^\dagger] = [Q, [C_i, B_j^\dagger]], \tag{2.24}$$

which follows from the representation (2.18) by means of the Jacobi identities, implies now that (2.21) requires

$$[C_i, B_j^\dagger] \text{ is an invertible matrix operator} \tag{2.25}$$

(cf. [6, 7]). Due to the definition (2.18) of the B-operators this condition requires that the set of C-operators $\{C_i\}$ is divided into two equally large sets, one with bosons and one with fermions, which in turn implies that the index i must run over an even number.

The above approach to a representation V_S of the BRST cohomology requires us to find a maximal irreducible set of operator doublets $\{D_r\}$ satisfying (2.19). (We shall call such a set a complete set of doublets.) V_S is then determined by conditions (2.17)-(2.18) *i.e.*

$$D_r|s\rangle = 0 \tag{2.26}$$

At least in the case when there is a ghost number operator N satisfying

$$[N, Q] = Q \tag{2.27}$$

one may prove that

$$Q|s\rangle = 0 \tag{2.28}$$

will always be implied by (2.26) since one then has

$$Q = a^r D_r \tag{2.29}$$

In the above approach we expect that one always can arrange the doublets so that there exists a nilpotent coBRST operator ${}^*Q$ satisfying

$$C_i = [{}^*Q, B_i] \tag{2.30}$$

This implies by means of the Jacobi identities that

$$[C_i, C_j^\dagger] = [\,^*Q, [B_i, C_j^\dagger]] \tag{2.31}$$

Thus, if the singlet states $|s\rangle$ are coBRST invariant, *i.e.* satisfies (2.11), then $[C_i, C_j^\dagger]$ vanishes between the singlet states. It follows also that singlet operators A_i should satisfy

$$[Q, A_i] = [\,^*Q, A_i] = 0 \tag{2.32}$$

which implies

$$\begin{aligned} &[C_i, A_j] = [\,^*Q, [Q, [C_i, A_j]]] \\ &[B_i, A_j] = [Q, [\,^*Q, [B_i, A_j]]] \end{aligned} \tag{2.33}$$

i.e. these commutators vanish between singlet states. We expect that a coBRST charge defined by (2.30) and satisfying nilpotency is equivalent to a coBRST charge defined by (2.9). Hence, in this case (2.26) should be equivalent to $\triangle|s\rangle = 0$.

3 A general setting for formula (1.1).

Let as before Q be a nilpotent BRST charge operator defined on a nondegenerate inner product space V. Determine then a maximal elementary set of operator doublets $\{D_{(1)r}\}$ which are in involution. However, in distinction to the previous section we require now that $D_{(1)r}$ are *hermitian*. If we then determine singlet states by

$$D_{(1)r}|s\rangle_0 = 0 \tag{3.1}$$

it is clear that the solutions $|s\rangle_0$ cannot belong to an inner product space since (2.21) is not satisfied. In this case it is, however, natural to expect that there exists an equally large set of BRST doublets $\{D_{(2)r}\}$ whose elements also are hermitian and in involution satisfying

$$[D_{(1)r}, D_{(2)s}] \;\; \text{is an invertible matrix operator} \tag{3.2}$$

In fact, this is just another polarization of the unphysical operators. Due to the hermiticity of the doublets there involution relations satisfy

$$[D_{(l)r}, D_{(l)s}] = K_{(l)rs}{}^t D_{(l)t} = D_{(l)t} K^{\dagger\;t}_{(l)sr}, \quad l = 1, 2 \tag{3.3}$$

We assume now that we have two such dual sets of hermitian BRST doublets $D_{(1)r}$ and $D_{(2)r}$. It follows then that we may define two sets of singlet states $|s\rangle_0^{(1,2)}$ by

$$D_{(l)r}|s\rangle_0^{(l)} = 0, \quad l = 1, 2 \tag{3.4}$$

neither of which belong to an inner product space but whose bilinear form ${}_0^{(1)}\langle s|s\rangle_0^{(2)}$ might be finite.

We shall now prove that the singlet states $|s\rangle_0^{(l)}$ may be related to singlet states $|s\rangle^{(l)}$ on inner product spaces under certain conditions. This relation is given by

$$|s\rangle^{(l)} = e^{[Q, \psi_l]}|s\rangle_0^{(l)} \tag{3.5}$$

where ψ_l is a specific odd hermitian operator. Obviously (3.4) imply

$$D'_{(l)r}|s\rangle^{(l)}=0, \quad l=1,2 \tag{3.6}$$

where

$$D'_{(l)r}\equiv e^{[Q,\psi_l]}D_{(l)r}e^{-[Q,\psi_l]} \tag{3.7}$$

also are BRST doublets since $[Q,\psi_l]$ is BRST invariant. This relation imply

$$D'^{\dagger}_{(l)r}=e^{-[Q,\psi_l]}D_{(l)r}e^{[Q,\psi_l]}\neq D'_{(l)r} \tag{3.8}$$

which provides for the possibility to satisfy (2.21). A necessary condition for this is that $C_{(l)i}$ and $B_{(l)i}$ of $D_{(l)r}\equiv(C_{(l)i},B_{(l)i})$ each consists of half bosons and half fermions. This will therefore be assumed to be the case in the following. We propose then that ψ_l in (3.5) should be expressed in terms of all the C-operators in the dual set of doublets according to the formula

$$\begin{aligned}\psi_1&\equiv C^{(b)}_{(2)a}C^{(f)a}_{(2)}\\ \psi_2&\equiv C^{(b)}_{(1)a}C^{(f)a}_{(1)}\end{aligned} \tag{3.9}$$

where $C^{(b)}_{(l)a}$ and $C^{(f)}_{(l)a}$ are the bosonic and fermionic operators of $C_{(l)i}(\equiv(C^{(b)}_{(l)a},C^{(f)}_{(l)a}))$ and where the indices a, b are supposed to be raised and lowered by means of a constant symmetric metric. Eq. (3.9) is a hermitian expression if $C^{(b)}_{(l)a}$ and $C^{(f)}_{(l)a}$ commute which we assume (otherwise we have to symmetrize (3.9)). Eq. (3.9) implies

$$\begin{aligned}[][Q,\psi_1]&=C^{(b)}_{(2)a}B^{(b)a}_{(2)}-iB^{(f)}_{(2)a}C^{(f)a}_{(2)a}\\ [Q,\psi_2]&=C^{(b)}_{(1)a}B^{(b)a}_{(1)}-iB^{(f)}_{(1)a}C^{(f)a}_{(1)a}\end{aligned} \tag{3.10}$$

where we have introduced hermitian B-operators defined by

$$B^{(f)}_{(l)a}\equiv i[Q,C^{(b)}_{(l)a}], \quad B^{(b)}_{(l)a}\equiv[Q,C^{(f)}_{(l)a}] \tag{3.11}$$

Thus, the right-hand side of $[Q,\psi_l]$ involves the complete dual set of doublets.

We assume now that the commutator $[C_{(1)i},C_{(2)j}]$ vanish between the bilinear form of the states $|s\rangle^{(1)}$ and $|s\rangle^{(2)}$ (cf. the statement over (2.24) in the nonhermitian case given in the previous section) which means that (3.2) requires

$$[B_{(1)i},C_{(2)j}]\text{ and }[B_{(2)i},C_{(1)j}]\text{ are invertible matrices} \tag{3.12}$$

which due to the split into fermionic and bosonic operators in turn requires

$$\begin{aligned}&[B^{(b)}_{(1)a},C^{(b)}_{(2)b}],\ [B^{(b)}_{(1)aa},C^{(b)}_{(2)b}],\ [B^{(f)}_{(1)a},C^{(f)}_{(2)b}],\text{ and}\\ &[B^{(f)}_{(2)a},C^{(f)}_{(1)b}]\text{ are invertible matrix operators}\end{aligned} \tag{3.13}$$

The BRST doublets fall then into quartets. These properties imply now

$$\begin{aligned}[][[Q,\psi_{(1)}],D_{(1)r}]&=A_r{}^sD_{(2)s}=-D_{(2)s}A^{\dagger\,s}_r\\ [[Q,\psi_{(2)}],D_{(2)r}]&=B_r{}^sD_{(1)s}=-D_{(1)s}B^{\dagger\,s}_r\end{aligned} \tag{3.14}$$

where $A_r{}^s$ and $B_r{}^s$ are invertible matrix operators. It follows then that

$$\begin{aligned} D'_{(1)r} &= e^{[Q,\psi_1]} D_{(1)r} e^{-[Q,\psi_1]} = D_{(1)r} + A_r{}^s D_{(2)s} + O(D^2_{(2)}) \\ D'_{(2)r} &= e^{[Q,\psi_2]} D_{(2)r} e^{-[Q,\psi_2]} = D_{(2)r} + B_r{}^s D_{(1)s} + O(D^2_{(1)}) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} D'^{\dagger}_{(1)r} &= e^{[Q,\psi_1]} D_{(1)r} e^{-[Q,\psi_1]} = D_{(1)r} - A_r{}^s D_{(2)s} + O(D^2_{(2)}) \\ D'^{\dagger}_{(2)r} &= e^{[Q,\psi_2]} D_{(2)r} e^{-[Q,\psi_2]} = D_{(2)r} - B_r{}^s D_{(1)s} + O(D^2_{(1)}) \end{aligned} \quad (3.16)$$

where $O(D^2_{(l)})$ denotes nonlinear terms in the $D_{(l)}$'s. The fact that the matrix operators $A_r{}^s$ and $B_r{}^s$ are nonsingular implies that $\{D'_{(1)r}\}$, $\{(D'_{(1)r})^{\dagger}\}$ as well as $\{D'_{(2)r}\}$ $\{(D'_{(2)r})^{\dagger}\}$ are algebraically independent sets and constitute generalized BRST quartets, *i.e.* the matrix operators $[D'_{(1)r}, (D'_{(1)s})^{\dagger}]$ as well as $[D'_{(2)r}, (D'_{(2)s})^{\dagger}]$ are nonsingular. Since $\{D_{(1)r}\}$ and $\{D_{(2)r}\}$ are complete sets the BRST doublets $\{D'_{(1)r}\}$ and $\{D'_{(2)r}\}$ are also complete sets. This implies that the physical states $|s\rangle^{(1,2)}$ in (3.5) each belongs to a nondegenerate physical state space representing the BRST cohomology. The assertion about (3.5) is then proved.

4 A simple example.

In order to get some insight into the above general structure of general BRST quantization we consider the simplest possible example when the hermitian BRST doublets $(C^{(b)}_{(l)a}, C^{(f)}_{(l)a}, B^{(b)}_{(l)a}, B^{(f)}_{(l)a})$ are completely elementary, *i.e.* the case when the only nonzero commutators are given by

$$\begin{aligned} [C^{(b)}_{(1)a}, B^{(b)b}_{(2)}] &= i\delta^a_b = [C^{(b)}_{(2)a}, B^{(b)b}_{(1)}] \\ [C^{(f)}_{(1)a}, B^{(f)b}_{(2)}] &= \delta^a_b = [C^{(f)}_{(2)a}, B^{(f)b}_{(1)}] \end{aligned} \quad (4.1)$$

In this case the coBRST charge operator is given by

$$^*Q = C^{(b)}_{(1)a} C^{(f)a}_{(2)} + C^{(b)}_{(2)a} C^{(f)a}_{(1)} \quad (4.2)$$

which implies

$$\begin{aligned} [^*Q, B^{(b)}_{(l)a}] &= iC^{(f)}_{(l)a} \\ [^*Q, B^{(f)}_{(l)a}] &= C^{(b)}_{(l)a} \end{aligned} \quad (4.3)$$

The BRST charge itself is of the form

$$Q = B^{(b)}_{(1)a} B^{(f)a}_{(2)} + B^{(b)}_{(2)a} B^{(f)a}_{(1)} \quad (4.4)$$

with the properties

$$\begin{aligned} [Q, C^{(b)}_{(l)a}] &= -iB^{(f)}_{(l)a} \\ [Q, C^{(f)}_{(l)a}] &= B^{(b)}_{(l)a} \end{aligned} \quad (4.5)$$

Notice that both Q and ${}^*Q$ are hermitian.

Consider now the antihermitian operator

$$R \equiv -iB^{(b)a}_{(1)}C^{(b)}_{(2)a} - iB^{(b)a}_{(2)}C^{(b)}_{(1)a} + B^{(f)a}_{(1)}C^{(f)}_{(2)a} + B^{(f)a}_{(2)}C^{(f)}_{(1)a} \tag{4.6}$$

It satisfies

$$[R, B^a] = B^a, \quad [R, C_a] = -C_a \tag{4.7}$$

which implies

$$[R, Q] = 2Q, \quad [R, {}^*Q] = -2\,{}^*Q \tag{4.8}$$

R may be split into two pieces

$$R = R_1 + R_2 \tag{4.9}$$

such that if the conjugate pair $(B^{(b)a}_{(2)},\ C^{(b)}_{(1)a})$ is in R_1, then the conjugate pair $(B^{(f)a}_{(1)},\ C^{(f)}_{(2)a})$ is in R_2 or vice versa. Then R_i satisfies

$$[R_i, Q] = Q, \quad [R_i, {}^*Q] = -\,{}^*Q \quad i = 1, 2 \tag{4.10}$$

Thus, R_1 (or R_2) properly chosen may be identified with the ghost number operator N in (2.27) if we have no operators with larger ghost number than ± 1. This implies then that half of the BRST doublets are ghosts. (If there are operators with larger ghost number, then more than half of the doublets have nonzero ghost number.)

Consider now the BRST laplacian $\triangle = [{}^*Q, Q]$. It is here given by

$$\begin{aligned}\triangle &= B^{(b)a}_{(1)}C^{(b)}_{(2)a} + C^{(b)a}_{(1)}B^{(b)}_{(2)a} - iB^{(f)a}_{(1)}C^{(f)}_{(2)a} + iC^{(f)a}_{(1)}B^{(f)}_{(2)a} = \\ &= B^{(b)a}_{(2)}C^{(b)}_{(1)a} + C^{(b)a}_{(2)}B^{(b)}_{(1)a} - iB^{(f)a}_{(2)}C^{(f)}_{(1)a} + iC^{(f)a}_{(2)}B^{(f)}_{(1)a}\end{aligned} \tag{4.11}$$

It is clear that

$$\triangle|s\rangle_0 = 0 \tag{4.12}$$

implies either

$$D_{(1)r}|s\rangle_0 = 0, \text{ or } D_{(2)r}|s\rangle_0 = 0 \tag{4.13}$$

(One has to choose the original state space such that either of these possibilities allow for solutions.) These two possibilities are also equally well implied by

$$Q|s\rangle_0 = {}^*Q|s\rangle_0 = 0 \tag{4.14}$$

The odd hermitian operator ψ_l in the formula (3.5) may now be chosen to be

$$\psi_l = [{}^*Q, F_l] \tag{4.15}$$

where

$$F_1 = B^{(f)a}_{(2)}C^{(f)}_{(2)a}, \quad F_2 = B^{(f)a}_{(1)}C^{(f)}_{(1)a} \tag{4.16}$$

These expressions are antihermitian and reproduce (3.9) and satisfy

$$[\triangle, F_l] = 0 \tag{4.17}$$

which is equivalent to

$$[Q, [{}^*Q, F_l]] = [{}^*Q, [Q, F_l]] \tag{4.18}$$

and implies for the inner product states in (3.5) that they also satisfies (4.14) *i.e.*

$$Q|s\rangle^{(l)} = {}^*Q|s\rangle^{(l)} = 0, \quad l = 1, 2 \tag{4.19}$$

The nonhermitian doublets defined by (3.7) become here

$$\begin{aligned} C'^{(b)}_{(1)a} &= C^{(b)}_{(1)a} - iC^{(b)}_{(2)a}, \quad C'^{(f)}_{(1)a} = C^{(f)}_{(1)a} + iC^{(f)}_{(2)a} \\ B'^{(b)}_{(1)a} &= B^{(b)}_{(1)a} - iB^{(b)}_{(2)a}, \quad B'^{(f)}_{(1)a} = B^{(f)}_{(1)a} + iB^{(f)}_{(2)a} \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} C'^{(b)}_{(2)a} &= -i(C'^{(b)}_{(1)a})^\dagger, \quad C'^{(f)}_{(2)a} = i(C'^{(f)}_{(1)a})^\dagger \\ B'^{(b)}_{(2)a} &= i(B'^{(b)}_{(1)a})^\dagger, \quad B'^{(f)}_{(2)a} = -i(B'^{(f)}_{(1)a})^\dagger \end{aligned} \tag{4.21}$$

Thus, in this case $\{D'_{(1)r},\ (D'_{(1)r})^\dagger\}$ as well as $\{D'_{(2)r},\ (D'_{(2)r})^\dagger\}$ are exactly the same sets. The BRST charge and the coBRST charge have the following form in terms of these nonhermitian doublets

$$\begin{aligned} Q &= \frac{i}{2}\left(B^{\dagger(b)}_a B^{(f)a} - B^{\dagger(f)}_a B^{(b)a}\right) \\ {}^*Q &= \frac{i}{2}\left(C^{\dagger(f)}_a C^{(b)a} - C^{\dagger(b)}_a C^{(f)a}\right) \end{aligned} \tag{4.22}$$

where we have made use of the short-hand notation

$$B^{(b,f)}_a \equiv B'^{(b,f)}_{(1)a}, \quad C^{(b,f)}_a \equiv C'^{(b,f)}_{(1)a} \tag{4.23}$$

We are now going to apply the above general properties to the BFV formulation of general gauge theories.

5 Gauge theories of arbitrary rank. The minimal sector.

Consider a classical theory whose Hamiltonian formulation is defined on a phase space Γ of dimension $2n$. It contains $m \leq n$ algebraically independent first class constraints

$$\theta_a = 0 \tag{5.1}$$

The constraint variables θ_a are assumed to be real with Grassmann parity $\varepsilon_a (= 0, 1)$. Their algebra in terms of the Poisson bracket on Γ is of the form

$$\{\theta_a, \theta_b\} = U_{ab}{}^c \theta_c \tag{5.2}$$

where ${U_{ab}}^c$ are arbitrary real functions on Γ consistent with the Jacobi identities. After quantization θ_a and ${U_{ab}}^c$ are turned into hermitian operators satisfying a commutator algebra of the form

$$[\theta_a, \theta_b] = i\frac{1}{2}\left(\theta_c {U_{ab}}^c + {U_{ab}}^c \theta_c\right) + \dots \tag{5.3}$$

where the precise form of the right-hand side depends on the quantization prescriptions for θ_a and ${U_{ab}}^c$. Anyway, for whatever choice made it is obvious that the Dirac quantization

$$\theta_a|\,\rangle = 0 \tag{5.4}$$

is not consistent if ${U_{ab}}^c$ are nontrivial operators which does not commute with θ_a in (5.3). As Dirac writes on page 70 in [13] "when we go over to the quantum theory we must insist that the coefficients $[{U_{ab}}^c]$ are on the left" in (5.3). A precise solution of this dilemma is provided by the BRST quantization.

We introduce therefore hermitian generalized Faddeev-Popov ghosts $\mathcal{C}^a$ and their conjugate momenta $\mathcal{P}_a$ with Grassmann parity $\varepsilon_a + 1$ satisfying

$$[\mathcal{C}^a, \mathcal{P}_b] = i\delta^a_b, \quad \mathcal{P}^\dagger_a = -(-1)^{\varepsilon_a}\mathcal{P}_a \tag{5.5}$$

The BFV-BRST charge operator for the above model is then given by [14, 15, 16, 17]

$$\Omega = \sum_{i=0}^{N} \Omega_i \tag{5.6}$$

where

$$\Omega_0 \equiv \mathcal{C}^a\theta_a, \quad \Omega_i \equiv \Omega^{b_1\cdots b_i}_{a_1\cdots a_{i+1}}(\mathcal{C}^{a_1}\cdots\mathcal{C}^{a_{i+1}}\mathcal{P}_{b_1}\cdots\mathcal{P}_{b_i})_{Weyl}, \quad i = 1,\dots,N \tag{5.7}$$

where "Weyl" indicates that the ghosts are Weyl ordered. N may be infinite provided the infinite sum in (5.6) makes sense. Ω is required to be hermitian which in turn requires Ω_i to be hermitian as well. According to ref.[17] there always exists nilpotent hermitian expressions of the form (5.6) and these solutions determine the precise form of the algebra (5.3).

Now a hermitian and nilpotent Ω may also be written in a $\mathcal{C}\mathcal{P}$-ordered form [15, 16, 17]

$$\Omega = \sum_{i=0}^{N} \Omega'_i, \quad \Omega'_0 \equiv \mathcal{C}^a\theta'_a,$$
$$\Omega'_i \equiv {\Omega'}^{b_1\cdots b_i}_{a_1\cdots a_{i+1}}\mathcal{C}^{a_1}\cdots\mathcal{C}^{a_{i+1}}\mathcal{P}_{b_1}\cdots\mathcal{P}_{b_i}, \quad i = 1,\dots,N \tag{5.8}$$

and in this case nilpotency requires the algebra

$$[\theta'_a, \theta'_b] = i{U'_{ab}}^c\theta'_c \tag{5.9}$$

which always allow for a consistent Dirac quantization given by

$$\theta'_a|\,\rangle = 0 \tag{5.10}$$

The structure functions ${U'_{ab}}^c$ in (5.9) are given by

$${U'_{ab}}^c = 2(-1)^{\varepsilon_a}{\Omega'_{ab}}^c \tag{5.11}$$

Identifying the expressions (5.6) and (5.8) one may always express θ_a, $\Omega^{b_1\cdots b_i}_{a_1\cdots a_{i+1}}$ in terms of θ'_a, $\Omega'^{b_1\cdots b_i}_{a_1\cdots a_{i+1}}$ or vice versa. One will then find relations of the type

$$\theta'_a = \theta_a + \sum_{k=1}^{N} i^k \Omega_{ab_1\cdots b_k}{}^{b_1\cdots b_k} \tag{5.12}$$

$$\Omega'_{ab}{}^{c} = \Omega_{ab}{}^{c} + \sum_{k=1}^{N} i^k \Omega_{abb_1\cdots b_k}{}^{cb_1\cdots b_k} \tag{5.13}$$

which implies that θ'_a in general are not hermitian. Exceptions are *e.g.* gauge theories invariant under unimodular Lie groups.

We may also choose a $\mathcal{PC}$-ordered Ω:

$$\begin{aligned} &\Omega = \sum_{i=0}^{N} \Omega''_i, \quad \Omega''_0 \equiv \theta''_a \mathcal{C}^a, \\ &\Omega''_i \equiv \Omega''^{b_1\cdots b_i}_{a_1\cdots a_{i+1}} \mathcal{P}_{b_1}\cdots\mathcal{P}_{b_i}\mathcal{C}^{a_1}\cdots\mathcal{C}^{a_{i+1}}, \quad i=1,\ldots,N \end{aligned} \tag{5.14}$$

and in this case $\Omega^2 = 0$ yields

$$[\theta''_a, \theta''_b] = i\theta''_c U''_{ab}{}^{c} \tag{5.15}$$

where

$$\theta''_a = (\theta'_a)^\dagger, \quad U''_{ab}{}^{c} = (-1)^{\varepsilon_a\varepsilon_b}(U'_{ab}{}^{c})^\dagger \tag{5.16}$$

These $\mathcal{CP}$- and $\mathcal{PC}$-ordered forms of Ω will be used in the following.

We look now for simple solutions of the BRST condition

$$\Omega|ph\rangle = 0 \tag{5.17}$$

relaxing for the moment the condition that $|ph\rangle$ should belong to an inner product space. We shall then make use of consistent auxiliary conditions [11, 12] eventually expressed in terms of hermitian BRST doublets in involution. Following [18] one may *e.g.* look for solutions which have no ghost dependence[2]. This may be done by means of a ghost fixing of the form [11]

$$g_a|ph\rangle = 0, \quad a = 1,\ldots,n \tag{5.18}$$

where g_a are m independent linear expressions of the ghosts $\mathcal{C}^a$ and $\mathcal{P}_a$. Consistency requires here that (5.18) must be accompanied by

$$[\Omega, g_a]|ph\rangle = 0, \quad a = 1,\ldots,m \tag{5.19}$$

and that g_a and $[\Omega, g_a]$ must satisfy a closed algebra with all coefficients to the left. The latter requires that Ω has a specific form except for the following two cases:

$$(1)\ g_a = \mathcal{C}^a, \quad (2)\ \ g_a = \mathcal{P}_a \tag{5.20}$$

[2]In a consistent gauge theory on inner product spaces the physical states should not contain any ghost excitations in order to have positive norms

In case (1) we have

$$[\Omega, \mathcal{C}^a] = \sum_{i=0}^{N} \Omega''^{b_1 \cdots b_i}_{a_1 \cdots a_{i+1}} [\mathcal{P}_{b_1} \cdots \mathcal{P}_{b_i}, \mathcal{C}^a] \mathcal{C}^{a_1} \cdots \mathcal{C}^{a_{i+1}} (-1)^{s_i},$$

$$s_i \equiv (\varepsilon_a + 1)(i + 1 + \sum_{k=1}^{i+1} \varepsilon_{a_k}) \tag{5.21}$$

where the $\mathcal{PC}$-ordered Ω (5.15) is used. In this case $\mathcal{C}^a$ and $[\Omega, \mathcal{C}^a]$ trivially satisfy a closed algebra. (Notice that $[\Omega, \mathcal{C}^a]$ satisfy a closed algebra among themselves only for theories of rank $0, 1$.)

In case (2) we have

$$[\Omega, \mathcal{P}_a] = \sum_{i=0}^{N} \Omega'^{a_1 \cdots a_i}_{b_1 \cdots b_{i+1}} [\mathcal{C}^{b_1} \cdots \mathcal{C}^{b_{i+1}}, \mathcal{P}_a] \mathcal{P}_{a_1} \cdots \mathcal{P}_{a_i} (-1)^{s_i} \tag{5.22}$$

where we use the more suitable $\mathcal{CP}$-ordered Ω (5.8). Also here do $\mathcal{P}_a$ and $[\Omega, \mathcal{P}_a]$ satisfy a closed algebra without any restrictions on Ω. The physical states may either be chosen to satisfy

$$\mathcal{C}^a |ph\rangle = 0 \tag{5.23}$$

or

$$\mathcal{P}_a |ph\rangle = 0, \quad [\Omega, \mathcal{P}_a] |ph\rangle = \theta'_a |ph\rangle = 0 \tag{5.24}$$

Notice that these conditions by themselves automatically imply the BRST condition (5.17) since

$$\Omega = A_a \mathcal{C}^a = \mathcal{C}^a \theta'_a + K^a \mathcal{P}_a \tag{5.25}$$

Thus, in case (1) we are led to the trivial ghost fixed solutions of (5.23) and in case (2) we are led to the consistent Dirac quantization (5.24).

A crucial question is now whether or not the constraint operators in (5.23) and (5.24) belong to BRST doublets. $\mathcal{P}_a$ and $[\Omega, \mathcal{P}_a]$ are BRST doublets provided $[\Omega, \mathcal{P}_a]$ represent m algebraic independent operators, which is the case if we have an irreducible gauge theory. In a reducible gauge theory there are linear combinations of $\mathcal{P}_a$ which are genuine physical operators which means that there are more genuine physical states than those determined by (5.24). Only irreducible gauge theories are considered here. The reducible case will be considered in section 8.

The conditions (5.23) are obtained from BRST doublets provided there exist m independent hermitian operators χ^a with ghost number zero satisfying

$$[\Omega, \chi^a] = i M^a_b \mathcal{C}^b \tag{5.26}$$

where M^a_b is a nonsingular matrix operator in the sense that $[\Omega, \chi^a]|ph\rangle = 0$ imply (5.23). In this case

$$\chi^a |ph\rangle = 0 \tag{5.27}$$

will imply (5.23), and (5.23) will allow for (5.27) [11]. Consistency requires χ^a and $[Q, \chi^a]$ to satisfy a closed algebra with all coefficients to the left. Since χ^a are naturally chosen to be independent of $\mathcal{C}^a$ this implies that they must be in involution:

$$[\chi^a, \chi^b] = V_c^{ab}\chi^c \tag{5.28}$$

χ^a represent unphysical degrees of freedom and they may be viewed as gauge fixing operators to the gauge generators $[\Omega, \mathcal{P}_a]$. Notice that

$$[\chi^b, [\Omega, \mathcal{P}_a]] = (-1)^{\varepsilon_b} {M'}^b_{\ a} \tag{5.29}$$

where ${M'}^b_{\ a}$ is given by

$${M'}^b_{\ a} = M^b_{\ a} + (-1)^{\varepsilon_a\varepsilon_c+\varepsilon_a+\varepsilon_c} i[M^b_{\ c}, \mathcal{P}_a]\mathcal{C}^c + [\Omega, [\chi^b, \mathcal{P}_a]] \tag{5.30}$$

which is obtained from the Jacobi identities. (The last term is zero if χ^a is independent of $\mathcal{C}^a$.) One may easily convince oneself that (5.27) yields (5.23) even if $M^a_{\ b}$ in (5.26) is replaced by ${M'}^a_{\ b}$.

If there is no gauge fixing operator χ^a satisfying the above conditions then there exist genuine physical operators (not belonging to any BRST doublets) with positive ghost number, which in turn means that there are other physical states than those determined by (5.23). Here we shall always assume that there exist gauge fixing operators χ^a satisfying the above conditions and that the gauge theory is irreducible. In this case the solutions of

$$\mathcal{C}^a|ph\rangle = \chi^a|ph\rangle = 0 \tag{5.31}$$

or

$$\mathcal{P}_a|ph\rangle' = [Q, \mathcal{P}_a]|ph\rangle' = 0 \tag{5.32}$$

each represent the genuine physical degrees of freedom, *i.e.* they are singlet states. In fact we have arrived at the general setting given in section 3: The conditions (5.31)-(5.32) may be expressed in terms of the following two dual sets of *hermitian* BRST doublets

$$D_{(1)r} \equiv \{\chi^a, i^{\varepsilon_a+1}[\Omega, \chi^a]\}, \quad D_{(2)r} \equiv \{i^{\varepsilon_a+1}\mathcal{P}_a, i[\Omega, \mathcal{P}_a]\} \tag{5.33}$$

For these we require that $[D_{(1)r}, D_{(2)s}]$ is an invertible matrix operator. Following section 3 we require also that the C-operators $\mathcal{P}_a$ and χ^a essentially commute so that this implies

$$\begin{array}{c}[\chi^a, [\Omega, \mathcal{P}_b]] \text{ and } [[\Omega, \chi^a], \mathcal{P}_b] \\ \text{are invertible matrix operators}\end{array} \tag{5.34}$$

Indeed this condition is satisfied since we have already shown that ${M'}^a_{\ b}$ in (5.30) is invertible when χ^a satisfies the doublet condition (5.26), *i.e.* when $M^a_{\ b}$ in (5.26) is invertible.

6 The nonminimal sector.

Since our goal is a BRST quantization on inner product spaces where the genuine physical states have ghost number zero in a consistent theory [19], we cannot in general make use of the solutions of (5.23) or (5.24) in expressions like (1.1) in the introduction since these

solutions have ghost number $m/2$ and $-m/2$ respectively. In order for such solutions always to have ghost number zero we need to introduce dynamical Lagrange multipliers and antighosts into the theory. The resulting extended BFV-BRST charge is then given by[3].

$$Q = \Omega + \bar{\mathcal{P}}^a \pi_a \tag{6.1}$$

where $\bar{\mathcal{P}}^a$ are conjugate momenta to the m antighosts $\bar{\mathcal{C}}_a$, and where π_a are conjugate momenta to the m Lagrange multipliers λ^a. Their hermiticity properties and Grassmann parities are

$$\begin{aligned} &\bar{\mathcal{P}}^{a\dagger} = \bar{\mathcal{P}}^a, \quad \bar{\mathcal{C}}_a^\dagger = -(-1)^{\varepsilon_a} \bar{\mathcal{C}}_a, \quad \pi_a^\dagger = \pi_a, \quad \lambda^{a\dagger} = (-1)^{\varepsilon_a} \lambda^a \\ &\varepsilon(\bar{\mathcal{P}}_a) = \varepsilon(\bar{\mathcal{C}}^a) = \varepsilon_a + 1, \quad \varepsilon(\pi_a) = \varepsilon(\lambda^a) = \varepsilon_a \end{aligned} \tag{6.2}$$

and they satisfy the commutation relations (the nonzero part)

$$|\bar{\mathcal{C}}_a, \bar{\mathcal{P}}^b] = i\delta_a^b, \quad [\lambda^a, \pi_b] = i\delta_b^a \tag{6.3}$$

Notice that only the ghosts and antighosts carry nonzero ghost numbers:

$$gh(\mathcal{C}) = -gh(\mathcal{P}) = gh(\bar{\mathcal{P}}) = -gh(\bar{\mathcal{C}}) = 1 \tag{6.4}$$

In this extended case we need further conditions in order to fix the ghost dependence of the physical states. The appropriate generalization of cases (1) and (2) in the previous section is then

$$(1) \quad \mathcal{C}^a |ph\rangle_{(1)} = \bar{\mathcal{C}}_a |ph\rangle_{(1)} = 0, \quad \pi_a |ph\rangle_{(1)} = 0 \tag{6.5}$$

$$(2) \quad \mathcal{P}_a |ph\rangle_{(2)} = \bar{\mathcal{P}}^a |ph\rangle_{(2)} = 0, \quad [Q, \mathcal{P}_a] |ph\rangle_{(2)} = 0 \tag{6.6}$$

In fact these conditions are the only consistent extensions of the previous cases which make the physical states have ghost number zero (such conditions were considered in [21, 11]). Notice that (6.5) and (6.6) are sufficient to make $|ph\rangle_{(1,2)}$ BRST invariant. However, as before they do not completely fix the physical states to a representation of the genuine physical degrees of freedom. In order to reduce $|ph\rangle_{(1,2)}$ to singlet states and comply with section 3 we have to impose the gauge fixing conditions

$$\chi^a |ph\rangle_{(1)} = 0, \quad \Lambda^a |ph\rangle_{(2)} = 0 \tag{6.7}$$

where the hermitian gauge fixing operators χ^a and Λ^a have ghost number zero and Grassmann parity ε_a, and are required to satisfy the conditions

$$\begin{aligned} &[Q, \chi^a] |ph\rangle_{(1)} = 0 \ \Rightarrow \ \mathcal{C}^a |ph\rangle_{(1)} = 0 \\ &[Q, \Lambda^a] |ph\rangle_{(2)} = 0 \ \Rightarrow \ \bar{\mathcal{P}}^a |ph\rangle_{(2)} = 0 \end{aligned} \tag{6.8}$$

when $\bar{\mathcal{C}}^a |ph\rangle_{(1)} = \pi_a |ph\rangle_{(1)} = 0$ and $\mathcal{P}_a |ph\rangle_{(2)} = [Q, \mathcal{P}_a] |ph\rangle_{(2)} = 0$, and the condition that the following sets of hermitian BRST doublets

$$\begin{aligned} &D_{(1)r} \equiv \{\chi^a, i^{\varepsilon_a+1}[Q, \chi^a]; i^{\varepsilon_a} \bar{\mathcal{C}}_a, i(-1)^{\varepsilon_a}[Q, \bar{\mathcal{C}}_a] = \pi_a\}, \\ &D_{(2)r} \equiv \{\Lambda^a, i^{\varepsilon_a+1}[Q, \Lambda^a]; i^{\varepsilon_a+1} \mathcal{P}_a, i[Q, \mathcal{P}_a]\} \end{aligned} \tag{6.9}$$

[3] Our notations are in accordance with those of [10-12], [14] except for the interchange $\mathcal{P} \leftrightarrow \bar{\mathcal{P}}$

each must satisfy a closed algebra with all coefficients to the left. Furthermore the BRST doublets (6.9) should constitute a set of generalized BRST quartets, *i.e.* $[D_{(1)r}, D_{(2)s}]$ must be an invertible matrix operator. As in section 3 we require the that the C-operators $(\chi^a, \bar{\mathcal{C}}_a, \mathcal{P}_a, \Lambda^a)$ essentially commute so that this condition is satisfied if

$$\begin{aligned}&[\chi^a, [Q, \mathcal{P}_b]], \quad [\bar{\mathcal{C}}_a, [Q, \Lambda_b]], \quad [\mathcal{P}_a, [Q, \chi^b]], \text{ and } [\pi_a, \Lambda^b]\\&\text{are invertible matrix operators}\end{aligned} \tag{6.10}$$

This is certainly satisfied if *e.g.* χ^a commutes with antighosts and Lagrange multipliers and satisfies the conditions for the minimal sector and if $\Lambda^a = \pm i^{\varepsilon_a}\lambda^a$. However, in addition there are more general solutions possible here. A nilpotent coBRST operator is here given by an expression like ${}^*Q = \chi^a \mathcal{P}_a + \bar{\mathcal{C}}_a \Lambda^a + \ldots$ which requires χ^a and Λ^a to be involution and independent of $\mathcal{C}^a$ and $\bar{\mathcal{P}}^a$.

7 The solutions on inner product spaces.

Now even though the above physical states, $|ph\rangle_{(1)}$ and $|ph\rangle_{(2)}$, have ghost number zero none of them belongs to an inner product space. In fact even if we assume the genuine physical variables to span an inner product space we still obtain undefined expressions like

$${}_{(l)}\langle ph|ph\rangle_{(l)} = 0 \cdot \infty, \quad l = 1, 2 \tag{7.1}$$

On the other hand, as in the minimal sector the bilinear forms ${}_{(1)}\langle ph|ph\rangle_{(2)}$ are in such a case finite and well defined. Now according to the results of section 3 the above states may be used to define BRST invariant states on inner product spaces. Formula (3.5) yields here

$$|ph, l\rangle = e^{[Q,\psi_l]}|ph\rangle_{(l)}, \; l = 1, 2 \tag{7.2}$$

where $|ph\rangle_{(l)}$ are the above physical states as defined in section 5, and where $\psi_{(l)}$ are the following hermitian fermionic gauge fixing operators with ghost number minus one

$$\psi_1 = i^{\varepsilon_a+1}\mathcal{P}_a\Lambda^a \tag{7.3}$$

$$\psi_2 = i^{\varepsilon_a+1}\bar{\mathcal{C}}_a\chi^a \tag{7.4}$$

where in turn Λ^a and χ^a are the gauge fixing operators of section 3 satisfying the conditions given there.

From section 3 we know that the states in (7.2) are formally inner product solutions for any consistent choice of the gauge fixing operators Λ^a and χ^a in (7.3) and (7.4). Furthermore, since $|ph\rangle_{(1,2)}$ need not satisfy the gauge fixing conditions (6.7) in order to be BRST invariant, one would naively expect the solutions $|ph, l\rangle$ (7.2) to be independent of the gauge fixing operators Λ^a and χ^a for $l = 1$ and $l = 2$ respectively. In particular one would expect the norms of $|ph, l\rangle$ to be independent of the choice of gauge fixing Λ^a and χ^a. Now, this is only true for certain classes of gauge fixings as will be demonstrated below. However, the gauge independence within such classes may be illustrated by means of some simplifying assumptions. Consider $|ph, 1\rangle$ first. Here we have

$$\begin{aligned}\langle ph, 1|ph, 1\rangle &= {}_{(1)}\langle ph|e^{2[Q,\psi_1]}|ph\rangle_{(1)} =\\&= {}_{(1)}\langle ph|e^{2i^{\varepsilon_a+1}([Q,\mathcal{P}_a]\Lambda^a+[Q,\Lambda^a]\mathcal{P}_a])}|ph\rangle_{(1)}\end{aligned} \tag{7.5}$$

If we assume Λ^a to be of the form $\Lambda^a = \frac{1}{2}(-i)^{\varepsilon_a} X^a_{\ b}\lambda^b$ where $X^a_{\ b}$ is a nonsingular matrix operator with commuting elements and with no dependence on the Lagrange multipliers, and which furthermore commutes with $[Q, \mathcal{P}_a]$, then we have

$$\langle ph, 1|ph, 1\rangle = {}_{(1)}\langle ph|e^{i[Q,\mathcal{P}_a]X^a_{\ b}\lambda^b + \bar{\mathcal{P}}^b X^a_{\ b}\mathcal{P}_a]}|ph\rangle_{(1)} \tag{7.6}$$

The fact that $\bar{\mathcal{P}}^a$, $\mathcal{P}_a$ have opposite Grassmann parity to $[Q, \mathcal{P}_a]$ and λ^a together with the properties of $|ph\rangle_{(1)}$ imply now that the only dependence on $X^a_{\ b}$ in (7.6) is through the factor: $\det X^a_{\ b}/(\det X^a_{\ b}) = 1$. Thus, (7.6) is independent of $X^a_{\ b}$. For $|ph, 2\rangle$ we have

$$\begin{aligned}\langle ph, 2|ph, 2\rangle &= {}_{(2)}\langle ph|e^{2[Q,\psi_2]}|ph\rangle_{(2)} = \\ &= {}_{(2)}\langle ph|e^{2(-i)^{\varepsilon_a}(-\pi_a\chi^a + \bar{\mathcal{C}}_a M^a_{\ b}\mathcal{C}^b)}|ph\rangle_{(2)} \propto \int \delta(\chi^a)\det(M^a_{\ b})\,|\phi|^2\end{aligned} \tag{7.7}$$

where ϕ is a matter state which is a solution of the Dirac quantization. Thus, here we get under some simplifying assumptions a standard Faddeev-Popov type of expression which is at least locally independent of χ^a [22]. (The quantization rules of [2] must be used in (7.6) and (7.7).)

The physical states for the cases 1 and 2, $|ph, 1\rangle$ and $|ph, 2\rangle$, should span the same physical state space if they are obtained from a given original inner product state. From section 3 this requires the complex BRST doublets $D'_{(1)r}$ and $D'_{(2)r}$ to be algebraically related. However, such a relation must be nonlinear in general which makes the equivalence hard to demonstrate. On the other hand, for abelian gauge theories there exist simple gauge fixing conditions χ^a for which there are no nonlinear terms in (3.15) and (3.16). In this case one may explicitly show that $D'_{(1)r}$ and $D'_{(2)r}$ are equivalent. (In fact, this is the example given in section 4.) Notice that when an equivalence is established then we also have an explicit solution of the Dirac quantization given by

$$|ph\rangle_{(2)} = e^{-[Q,\psi_2]}e^{[Q,\psi_1]}|ph\rangle_{(1)} \tag{7.8}$$

It should be stressed that the physical state space is spanned by (7.2) for one specific choice of the gauge fixing operators Λ^a and χ^a in (7.3) and (7.4). Although there are whole classes of Λ^a and χ^a which yield physical states belonging to the same physical inner product space there always exist choices which do not. To demonstrate this consider *e.g.* (7.2) for Λ^a, χ^a and $-\Lambda^a$, $-\chi^a$, *i.e.*

$$|ph, l\rangle = e^{[Q,\psi_l]}|ph\rangle_{(l)}, \quad |ph, l\rangle' = e^{-[Q,\psi_l]}|ph\rangle_{(l)} \tag{7.9}$$

Obviously $|ph, l\rangle$ and $|ph, l\rangle'$ do not belong to the same inner product space since their inner product is undefined, $\langle ph, l|ph, l\rangle' = {}_{(l)}\langle ph|ph\rangle_{(l)} = 0\cdot\infty$. They are simply spanned by inequivalent bases which means that the corresponding original state spaces are also spanned by inequivalent bases. This implies that the choice of gauge fixing is related to the choice of an original inner product space from which the physical states are projected out. On the other hand, it seems as if one always may impose the condition that the physics of $|ph, l\rangle$ and $|ph, l\rangle'$ should be equivalent [1, 2] although they are projected from two different state spaces.

Now different choices of gauge fixing lead in general to equivalent state spaces. There is *e.g.* always a class of unitary equivalent choices for the gauge fixing operator ψ: Let U be a BRST invariant unitary operator with ghost number zero, *i.e.*

$$[Q, U] = 0, \quad [N, U] = 0, \quad U^\dagger U = UU^\dagger = \mathbf{1} \tag{7.10}$$

This implies

$$U|ph,l\rangle=e^{[Q,\psi_l']}U|ph\rangle_{(l)} \tag{7.11}$$

where

$$\psi_l'=U\psi_l U^\dagger \tag{7.12}$$

Thus, for those U for which $U|ph\rangle_{(l)}$ satisfies the same conditions as $|ph\rangle_{(l)}$, U transforms the gauge fixing operators within the same state space. An important example of such a transformation is

$$U=e^{\alpha[Q,\lambda^a\bar{\mathcal{C}}_a]} \tag{7.13}$$

where α is a real parameter. It satisfies (7.10) and since

$$[Q,\lambda^a\bar{\mathcal{C}}_a]=i\lambda^a\pi_a-i(-1)^{\varepsilon_a}\bar{\mathcal{P}}^a\bar{\mathcal{C}}_a \tag{7.14}$$

we have

$$U|ph\rangle_1=|ph\rangle_1 \tag{7.15}$$

when $|ph\rangle_1$ satisfies (6.5), and

$$U|ph,\lambda,\pi\rangle_2=|ph,e^{-\alpha}\lambda,e^{\alpha}\pi\rangle_2 \tag{7.16}$$

when $|ph\rangle_2$ satisfies (6.6). (The Lagrange multiplier dependence in $|ph\rangle_2$ represents unphysical gauge degrees of freedom. It may be fixed by the condition (6.7).) For the gauge fixing fermions (7.3) and (7.4) we simply get a scaling

$$\psi_1'=U\psi_1U^\dagger=e^{-\alpha}\psi_1,\quad \psi_2'=U\psi_2U^\dagger=e^{\alpha}\psi_2 \tag{7.17}$$

provided χ^a does not involve the Lagrange multipliers in other combinations than $\lambda^a\pi_b$, and provided Λ^a is linear in λ^a. Thus, we may always scale the exponents in (7.2) without affecting the norms and physical contents of the states [1, 2].

8 The general reducible case

The above results for the irreducible case may also be extended to the general reducible case. Let us consider a general L-stage reducible gauge theory. Here the basic gauge generators θ_{a_0}, $a_0=1,\ldots,m_0$, satisfy not only the involution relations but also the additional conditions [20]

$$\begin{aligned}&\theta_{a_0}Z^{a_0}_{1a_1}=0,\quad a_1=1,\ldots,m_1\\&Z^{a_{s-2}}_{s-1\,a_{s-1}}Z^{a_{s-1}}_{sa_s}=0,\quad a_s=1,\ldots,m_s,\ \ s=2,\ldots,L\end{aligned} \tag{8.1}$$

where

$$\operatorname{rank} Z^{a_{s-1}}_{sa_s}=\gamma_s(L),\quad \gamma_s(L)\equiv\sum_{s'=s}^{L}m_{s'}(-1)^{(s'-s)} \tag{8.2}$$

These relations imply that the m_0 constraint variables θ_{a_0} are dependent and only represent $\gamma_0(L) < m_0$ irreducible constraints. An invariant BRST charge requires us to introduce m_0 ghost variables to θ_{a_0} which is too many. On the other hand these extra ghosts may be compensated by the introduction of ghosts for ghosts. The resulting nilpotent BRST charge operator in the minimal sector involves then the following total set of ghosts and their conjugate momenta [20]:

$$\mathcal{C}_s^{a_s},\ \mathcal{P}_s^{a_s},\quad s=0,\ldots,L;\ a_s=1,\ldots,m_s \tag{8.3}$$

which satisfy

$$[\mathcal{C}_s^{a_s},\mathcal{P}_{s'a_{s'}}]=i\delta_{ss'}\delta^{a_s}_{a_{s'}} \tag{8.4}$$

The BFV-BRST charge is of the form

$$\Omega=\mathcal{C}_0^{a_0}\theta_{a_0}+\sum_{s=0}^{L-1}\mathcal{C}_{s+1}^{a_{s+1}}Z^{a_s}_{a_{s+1}}\mathcal{P}_{sa_s}+\text{higher order terms in the ghosts} \tag{8.5}$$

The required ghost numbers and Grassmann parities are

$$\begin{aligned}&gh(\mathcal{C}_s^{a_s})=s+1,\quad gh(\mathcal{P}_{sa_s})=-(s+1)\\ &\varepsilon(\mathcal{C}_s^{a_s})=\varepsilon_{a_s}+s+1=\varepsilon(\mathcal{P}_{sa_s})\end{aligned} \tag{8.6}$$

where ε_{a_s} is defined by

$$\begin{aligned}&\varepsilon_{a_0}\equiv\varepsilon(\theta_{a_0})\\ &\varepsilon_{a_s}\equiv\varepsilon(Z^{a_s}_{a_{s+1}})-\varepsilon_{a_{s-1}},\quad s\geq 1\end{aligned} \tag{8.7}$$

θ_{a_0} and $\mathcal{C}_s^{a_s}$ are assumed to be hermitian which implies $(\mathcal{P}_{sa_s})^\dagger=(-1)^{\varepsilon_{a_s}}\mathcal{P}_{sa_s}$ from (8.4).

We try now to fix the ghost dependence of the physical states. As in the irreducible case there are only two sets of conditions which may be possible to impose in general. They are

$$(1)\qquad \mathcal{C}_s^{a_s}|ph\rangle_{(1)}=0,\quad s=0,\ldots,L;\ \ a_s=1,\ldots,m_s \tag{8.8}$$

$$(2)\qquad \mathcal{P}_{sa_s}|ph\rangle_{(2)}=0,\quad s=0,\ldots,L;\ \ a_s=1,\ldots,m_s \tag{8.9}$$

Consistency requires

$$(1)\qquad [\Omega,\mathcal{C}_s^{a_s}]|ph\rangle_{(1)}=0,\quad s=0,\ldots,L;\ \ a_s=1,\ldots,m_s \tag{8.10}$$

$$(2)\qquad [\Omega,\mathcal{P}_{sa_s}]|ph\rangle_{(2)}=0,\quad s=0,\ldots,L;\ \ a_s=1,\ldots,m_s \tag{8.11}$$

Eq. (8.10) is automatically satisfied (use a $\mathcal{PC}$-ordered Ω) while (8.11) only contains the following new nontrivial condition

$$[\Omega,\mathcal{P}_{0a_0}]|ph\rangle_{(2)}\equiv(\theta_{a_0}+\ldots)|ph\rangle_{(2)}=0 \tag{8.12}$$

in addition to (8.9) (use a $\mathcal{CP}$-ordered Ω)

Let us now look for consistent sets of BRST doublets. In case (2) we have

$$[\Omega, \mathcal{P}_{saQs}] = iZ^{a_{s-1}}_{a_s}\mathcal{P}_{s-1\,a_{s-1}} + (\;)\mathcal{P}^2 \tag{8.13}$$

which implies that not all $\mathcal{P}_{sa_s}$ are C-operators. In fact, since

$$\mathrm{rank} Z^{a_{s-1}}_{a_s} = \gamma_s(L) \tag{8.14}$$

only $\gamma_s(L)$ $\mathcal{P}_{sa_s}$-operators are C-operators. This is less than m_s since

$$\gamma_s(L) + \gamma_{s+1}(L) = m_s, \quad \gamma_L(L) = m_L \tag{8.15}$$

from (8.2). We define these C-operators to be

$$P_{sa_s} \equiv R^{b_s}_{sa_s}\mathcal{P}_{sb_s} \tag{8.16}$$

where $R^{b_s}_{sa_s}$ satisfies

$$\mathrm{rank} R^{b_s}_{sa_s} = \gamma_s(L), \quad \mathrm{rank} R^{b_s}_{sa_s} Z^{a_{s-1}}_{b_s} = \gamma_s(L) \quad \varepsilon(R^{b_s}_{sa_s}) = \varepsilon_{a_s} + \varepsilon_{b_s} \tag{8.17}$$

We also require $R^{b_s}_{sa_s}$ to be such that P_{sa_s} is hermitian. Now, due to (8.15) there are $\gamma_{s+1}(L)$ $\mathcal{P}_{sa_s}$-operators which are B-operators and from (8.13) they are given by $Z^{a_{s-1}}_{b_s}(+\cdots)\mathcal{P}_{sb_s}$. The total set of doublets in case (2) are then $\{P_{sa_s}; [\Omega, P_{sa_s}]\}$. Notice that they are equivalent to (8.9) and (8.12). The counting agrees since the total number of constraint operators are

$$2\sum_{s=0}^{L}\gamma_s(L) = \gamma_0(L) + \sum_{s=0}^{L} m_s \tag{8.18}$$

(For each s there are $\gamma_s(L)$ doublets, and there are m_s $\mathcal{P}_{sa_s}$-operators for each s plus $\gamma_0(L)$ irreducible original constraints.)

For the ghost part (case (1)) we introduce hermitian gauge fixing operators $\chi^{a_s}_s$ satisfying

$$\begin{aligned} &[\Omega, \chi^{a_s}_s] = iK^{a_s}_{sb_s}\mathcal{C}^{b_s}_s \\ &gh(\chi^{a_s}_s) = s, \quad \mathrm{rank} K^{a_s}_{sb_s} = \gamma_s(L) \end{aligned} \tag{8.19}$$

Notice that $K^{a_0}_{0b_0}$ can only have rank $\gamma_0(L)$ since θ_{a_0} only involves $\gamma_0(L)$ independent components. Thus, $\chi^{a_0}_0$ involves only $\gamma_0(L)$ independent components and are as in the irreducible case also gauge fixing conditions to $[\Omega, P_{0a_0}]$. Since

$$[\Omega, \mathcal{C}^{a_s}_s] = \mathcal{C}^{a_{s+1}}_{s+1} Z^{a_s}_{a_{s+1}} \tag{8.20}$$

it follows that we may choose

$$\chi^{a_s}_s \equiv \omega^{a_s}_{sa_{s-1}}\mathcal{C}^{a_{s-1}}_{s-1}, \quad s \geq 1 \tag{8.21}$$

where

$$\mathrm{rank}\,\omega^{a_s}_{sa_{s-1}} = \gamma_s(L), \quad \mathrm{rank}\, Z^{a_{s-1}}_{b_s}\omega^{a_s}_{sa_{s-1}} = \gamma_s(L) \tag{8.22}$$

which implies that $\mathrm{rank} K^{a_s}_{sb_s} = \gamma_s(L)$ in (8.19). Obviously $(\chi^{a_s}_s, K^{a_s}_{sb_s}\mathcal{C}^{b_s}_s)$ constitute $\gamma_s(L)$ BRST doublets for each s. These variables are equivalent to $\mathcal{C}^{a_s}_s$ for all s plus $\chi^{a_0}_0$. (The

counting agrees due to the equality (8.18).) We conclude that in order to project out BRST singlets the ghost fixing (8.8) and (8.9) must be accompanied by the additional conditions

$$\begin{aligned}&(1)\quad \chi_0^{a_0}|ph\rangle_{(1)}=0,\quad a_0=1,\ldots,m_0\\&(2)\quad [\Omega,\mathcal{P}_{0a_0}]|ph\rangle_{(2)}=0,\quad a_0=1,\ldots,m_0\end{aligned}\tag{8.23}$$

We have ,thus, found the following two sets of BRST doublets

$$\begin{aligned}D_{(1)r}&=\{\chi_s^{a_s},[\Omega,\chi_s^{a_s}],\ a_s=1,\ldots,m_s;\ s=0,\ldots,L\}\\D_{(2)r}&=\{P_{sa_s},[\Omega,P_{sa_s}],\ \ a_s=1,\ldots,m_s;\ s=0,\ldots,L\}\end{aligned}\tag{8.24}$$

Each of them are required to be in involution, and with appropriate factors of i they may be chosen to be hermitian. We require now that they are dual sets of doublets so that they together form BRST quartets. In other words we require the matrix operator $[D_{(1)r},D_{(2)s}]$ to be invertible. As before we demand that $[\chi_s^{a_s},P_{s'a_{s'}}]$ essentially vanish so that it is sufficient to require

$$\begin{aligned}\text{rank}[P_{sa_s},[\Omega,\chi_{s'}^{b_{s'}}]]&=\gamma_s(L)\\\text{rank}[\chi_s^{a_s},[\Omega,P_{s'b_{s'}}]]&=\gamma_s(L)\end{aligned}\tag{8.25}$$

These conditions are satisfied by the conditions which we have already imposed. Notice that

$$\text{rank}[\chi_0^{a_0},[\Omega,P_{0b_0}]]=\gamma_0(L)\tag{8.26}$$

follows from $\text{rank}K^{a_0}_{0b_0}=\gamma_0(L)$ in (8.19) due to the Jacobi identities. That $[\chi_s^{a_s},P_{s'a_{s'}}]$ essentially vanishes requires the following connections between the matrix operators $\omega^{a_s}_{sb_{s-1}}$ and $R^{aQs}_{s\,b_s}$:

$$\omega^{a_s}_{sb_{s-1}}R^{b_{s-1}}_{s-1\,a_{s-1}}=0\tag{8.27}$$

We turn now to the nonminimal sector. In [20] it was shown that the correct form of the extended BRST charge is given by

$$Q=\Omega+\sum_{s=0}^{L}\pi_{sa_s}\bar{\mathcal{P}}_s^{a_s}+\sum_{s'=1}^{L}\sum_{s=s'}^{L}\pi^{s'}_{sa_s}\bar{\mathcal{P}}_s^{s'a_s}\tag{8.28}$$

where the new variables have the properties (π_{sa_s} and $\bar{\mathcal{P}}_s^{a_s}$ are chosen to be hermitian)

$$\begin{aligned}&[\lambda_s^{b_s},\pi_{ra_r}]=i\delta_{sr}\delta^{b_s}_{a_r},\quad[\bar{\mathcal{C}}_{sa_s},\bar{\mathcal{P}}_r^{b_r}]=i\delta_{sr}\delta^{b_r}_{a_s}\\&gh(\pi_{sa_s})=-gh(\lambda_s^{a_s})=-s,\\&gh(\bar{\mathcal{C}}_{sa_s})=-gh(\bar{\mathcal{P}}_s^{a_s})=-(s+1)\\&\varepsilon(\pi_{sa_s})=\varepsilon(\lambda_s^{a_s})=\varepsilon_{a_s}+s,\\&\varepsilon(\bar{\mathcal{C}}_{sa_s})=\varepsilon(\bar{\mathcal{P}}_s^{a_s})=\varepsilon_{a_s}+s+1\\&a_s=1,\ldots,m_s,\quad s=0,\ldots,L\end{aligned}\tag{8.29}$$

and ($\pi^{s'}_{sa_s}$ and $\bar{\mathcal{P}}^{s'a_s}_s$ are chosen to be hermitian)

$$\begin{aligned}
&[\lambda^{s'b_s}_s, \pi^{r'}_{ra_r}] = i\delta^{s'r'}\delta_{sr}\delta^{b_s}_{a_r}, \quad [\bar{\mathcal{C}}^{s'}_{sa_s}, \bar{\mathcal{P}}^{r'b_r}_r] = i\delta^{s'r'}\delta_{sr}\delta^{b_r}_{a_s} \\
&gh(\pi^{s}_{sa_s}) = -gh(\lambda^{s'a_s}_s) = -(s'-s), \\
&gh(\bar{\mathcal{C}}^{s'}_{sa_s}) = -gh(\bar{\mathcal{P}}^{s'a_s}_s) = -(s-s'+1) \\
&\varepsilon(\pi^{s'}_{sa_s}) = \varepsilon(\lambda^{s'a_s}_s) = \varepsilon_{a_s} + s - s', \\
&\varepsilon(\bar{\mathcal{C}}^{s'}_{sa_s}) = \varepsilon(\bar{\mathcal{P}}^{s'a_s}_s) = \varepsilon_{a_s} + s - s' \\
&a_s = 1, \ldots, m_s, \quad s = s', \ldots, L, \quad s' = 1, \ldots, L
\end{aligned} \tag{8.30}$$

All these new variables are unphysical. They may be grouped into additional BRST doublets, $(\bar{\mathcal{C}}, \pi)$ and $(\lambda, \bar{\mathcal{P}})$, which commute with the ones in the minimal sector. The question is now how to combine these doublets with the original ones in cases (1) and (2) so that the physical states will have ghost number zero. Below we give a simple algoritm how to split the doublets appropriately.

Guided by the irreducible case we first add $(\bar{\mathcal{C}}_{sa_s}, \pi_{sa_s})$ to $D_{(1)r}$ and $(\lambda^{a_s}_s, \bar{\mathcal{P}}^{a_s}_s)$ to $D_{(2)r}$, where $D_{(1,2)r}$ are defined in (8.24). We apply then the general formula (3.5) in section 3. We find then the hermitian gauge fixing operators (P_{sa_s} and $\chi^{a_s}_s$ are here chosen such that $\psi_{1,2}$ are hermitian)

$$\begin{aligned}
\psi_1 &= \sum_{s=0}^{L} P_{sa_s}\lambda^{a_s}_s \\
\psi_2 &= \sum_{s=0}^{L} \bar{\mathcal{C}}_{sa_s}\chi^{a_s}_s
\end{aligned} \tag{8.31}$$

Now since χ_s and P_s each only represents $\gamma_s(L)$ independent variables, $\gamma_{s+1}(L)$ of the λ_s and $\bar{\mathcal{C}}_s$ variables are not involved in (8.31). In order to introduce also them into the gauge fixing function we need auxiliary variables. The additional terms must have the form

$$\begin{aligned}
\triangle_1\psi_1 &= \sum_{s=0}^{L-1} P^1_{sa_s}\lambda^{a_s}_s, \quad gh(P^1_{sa_s}) = -(s+1) \\
\triangle_1\psi_2 &= \sum_{s=0}^{L-1} \bar{\mathcal{C}}_{sa_s}\chi^{1a_s}_s, \quad gh(\chi^{1a_s}_s) = s
\end{aligned} \tag{8.32}$$

where $\chi^{1a_s}_s$ and $P^1_{sa_s}$ each represents $\gamma_{s+1}(L)$ new degrees of freedom. Since we have only covariant variables at our disposal we define (cf. [20])

$$\begin{aligned}
\chi^{1a_{s-1}}_{s-1} &\equiv \bar{\omega}^{1a_{s-1}}_{s-1\,a_s}\lambda^{1a_s}_s, \quad \operatorname{rank}\bar{\omega}^{1a_{s-1}}_{s-1\,a_s} = \gamma_s(L) \\
P^1_{s-1\,a_{s-1}} &\equiv \bar{\mathcal{C}}^1_{sa_s}\sigma^{1a_s}_{sa_{s-1}}, \quad \operatorname{rank}\sigma^{1a_s}_{sa_{s-1}} = \gamma_s(L)
\end{aligned} \tag{8.33}$$

However, now we have introduced too many $\lambda^{1a_s}_s$ and $\bar{\mathcal{C}}^1_{sa_s}$ variables. We must therefore add

$$\begin{aligned}
\triangle_2\psi_1 &= \sum_{s=1}^{L-1} \bar{\mathcal{C}}^1_{sa_s}\chi^{2a_s}_s, \quad gh(\chi^{2a_s}_s) = s-1 \\
\triangle_2\psi_2 &= \sum_{s=1}^{L-1} P^2_{sa_s}\lambda^{1\,a_s}_s, \quad gh(P^2_{sa_s}) = -s
\end{aligned} \tag{8.34}$$

where

$$
\begin{aligned}
&\chi_{s-1}^{2a_{s-1}} \equiv \bar{\omega}_{s-1\,a_s}^{2a_{s-1}} \lambda_s^{2a_s}, \quad \operatorname{rank} \bar{\omega}_{s-1\,a_s}^{2\,a_{s-1}} = \gamma_s(L) \\
&P^2_{s-1\,a_{s-1}} \equiv \bar{\mathcal{C}}^2_{sa_s} \sigma^{2a_s}_{sa_{s-1}}, \quad \operatorname{rank} \sigma^{2\,a_s}_{sa_{s-1}} = \gamma_s(L)
\end{aligned} \tag{8.35}
$$

But with (8.34) we have introduced too many $\lambda_s^{2a_s}$ and $\bar{\mathcal{C}}^2_{sa_s}$ variables which forces us to add further terms. This procedure eventually stops and we arrive at the final formulas

$$
\begin{aligned}
\psi_1 = \; & \sum_{s=0}^{L} P_{sa_s} \lambda_s^{a_s} + \sum_{s=0}^{L-1} P^1_{sa_s} \lambda_s^{a_s} + \sum_{s=1}^{L-1} \bar{\mathcal{C}}^1_{sa_s} \chi_s^{2a_s} + \sum_{s=2}^{L-1} P^3_{sa_s} \lambda_s^{2a_s} + \\
& + \sum_{s=3}^{L-1} \bar{\mathcal{C}}^3_{sa_s} \chi_s^{4a_s} + \ldots = \sum_{s=0}^{L} P_{sa_s} \lambda_s^{a_s} + \\
& + \sum_{s'=0}^{[(L-1)/2]} \sum_{s=2s'+1}^{L} \left(\bar{\mathcal{C}}^{2s'+1}_{sa_s} \sigma^{2s'+1\,a_s}_{sa_{s-1}} \lambda_{s-1}^{2s'\,a_{s-1}} \right) + \\
& + \sum_{s'=1}^{[L/2]} \sum_{s=2s'}^{L} \left(\bar{\mathcal{C}}^{2s'-1}_{s-1\,a_{s-1}} \bar{\omega}^{2s'\,a_{s-1}}_{s-1\,a_s} \lambda_s^{2s'\,a_s} \right)
\end{aligned} \tag{8.36}
$$

$$
\begin{aligned}
\psi_2 = \; & \sum_{s=0}^{L} \bar{\mathcal{C}}_{sa_s} \chi_s^{a_s} + \sum_{s=0}^{L-1} \bar{\mathcal{C}}_{sa_s} \chi_s^{1a_s} + \sum_{s=1}^{L-1} P^2_{sa_s} \lambda_s^{1a_s} + \sum_{s=2}^{L-1} \bar{\mathcal{C}}^2_{sa_s} \chi_s^{3a_s} + \\
& + \sum_{s=3}^{L-1} P^4_{sa_s} \lambda_s^{3a_s} + \ldots = \sum_{s=0}^{L} \bar{\mathcal{C}}_{sa_s} \chi_s^{a_s} + \\
& + \sum_{s'=1}^{[L/2]} \sum_{s=2s'}^{L} \left(\bar{\mathcal{C}}^{2s'}_{sa_s} \sigma^{2s'\,a_s}_{sa_{s-1}} \lambda_{s-1}^{2s'-1\,a_{s-1}} \right) + \\
& + \sum_{s'=0}^{[(L-1)/2]} \sum_{s=2s'+1}^{L} \left(\bar{\mathcal{C}}^{2s'}_{s-1\,a_{s-1}} \bar{\omega}^{2s'+1\,a_{s-1}}_{s-1\,a_s} \lambda_s^{2s'+1\,a_s} \right)
\end{aligned} \tag{8.37}
$$

where

$$
\begin{aligned}
&\chi_{s-1}^{s'\,a_{s-1}} \equiv \bar{\omega}_{s-1\,a_s}^{s'\,a_{s-1}} \lambda_s^{s'\,a_s}, \quad \operatorname{rank} \bar{\omega}_{s-1\,a_s}^{2a_{s-1}} = \gamma_s(L) \\
&P^{s'}_{s-1\,a_{s-1}} \equiv \bar{\mathcal{C}}^{s'}_{sa_s} \sigma^{s'\,a_s}_{sa_{s-1}}, \quad \operatorname{rank} \sigma^{1a_s}_{sa_{s-1}} = \gamma_s(L) \\
&\lambda_s^{0\,a_s} \equiv \lambda_s^{a_s}, \quad \bar{\mathcal{C}}^0_{sa_s} \equiv \bar{\mathcal{C}}_{sa_s}
\end{aligned} \tag{8.38}
$$

Obviously the auxiliary variables in (8.30) are exactly what was needed to get the counting right. In fact, they could have been derived by this argument.

From the expressions (8.36) we notice that ψ_1 involves the additional C-operators $(\lambda^{2s'}, \bar{\mathcal{C}}^{2s'-1})$, $s' = 1, \ldots, [L/2]$, while ψ_2 involves $(\lambda^{2s'-1}, \bar{\mathcal{C}}^{2s'})$, $s' = 1, \ldots, [L/2]$. The final set of effective BRST doublets for cases (1) and (2) are then

$$
\begin{aligned}
&(1) \qquad \{(\chi_s^{a_s}, [\Omega, \chi_s^{a_s}]); (\bar{\mathcal{C}}^{2s'}_{sa_s}, \pi^{2s'}_{sa_s}); (\bar{\mathcal{P}}_s^{2s'-1\,a_s}, \lambda_s^{2s'-1\,a_s})\} \\
&(2) \qquad \{(P_{sa_s}, [\Omega, P_{sa_s}]); (\bar{\mathcal{C}}^{2s'-1}_{sa_s}, \pi^{2s'-1}_{sa_s}); (\bar{\mathcal{P}}_s^{2s'\,a_s}, \lambda_s^{2s'\,a_s})\}
\end{aligned} \tag{8.39}
$$

This implies that the singlet states $|ph\rangle_{(1,2)}$ are determined by the conditions

$$\begin{aligned}(1)\qquad &\chi_0^{a_0}|ph\rangle_{(1)} = \mathcal{C}_s^{a_s}|ph\rangle_{(1)} = \bar{\mathcal{C}}_{sa_s}|ph\rangle_{(1)} = \pi_{sa_s}|ph\rangle_{(1)} = \\ &= \bar{\mathcal{C}}_{sa_s}^{2s'}|ph\rangle_{(1)} = \pi_{sa_s}^{2s'}|ph\rangle_{(1)} = \bar{\mathcal{P}}_s^{2s'-1\,a_s}|ph\rangle_{(1)} = \lambda_s^{2s'-1\,a_s}|ph\rangle_{(1)} = 0 \end{aligned} \quad (8.40)$$

$$\begin{aligned}(2)\qquad &[\Omega, P_{0a_0}]|ph\rangle_{(2)} = \mathcal{P}_{sa_s}|ph\rangle_{(2)} = \bar{\mathcal{P}}_s^{a_s}|ph\rangle_{(2)} = \lambda_s^{a_s}|ph\rangle_{(2)} = \\ &= \bar{\mathcal{P}}_s^{2s'\,a_s}|ph\rangle_{(2)} = \lambda_s^{2s'\,a_s}|ph\rangle_{(2)} = \bar{\mathcal{C}}_{sa_s}^{2s'-1}|ph\rangle_{(2)} = \pi_{sa_s}^{2s'-1}|ph\rangle_{(2)} = 0 \end{aligned} \quad (8.41)$$

One may easily check that these conditions imply that $|ph\rangle_{(1)}$ and $|ph\rangle_{(2)}$ have ghost number zero. The physical states on an inner product space are then given by

$$|ph, l\rangle = e^{[Q,\psi_l]}|ph\rangle_{(l)}, \quad l = 1, 2 \qquad (8.42)$$

The reducible case on inner product spaces has also been treated in [23] and [24]. In [23] the linear case (perturbative unitarity) is treated and in [24] some traces of case (2) is given within the path integral formulation, the conditions (8.41) are *e.g.* imposed as boundary conditions (cf. [3]).

9 Conclusions

We have derived general formal solutions of BRST quantizations on inner product spaces by means of a generalized quartet mechanism. For this our starting point was that the physical states, which constitute a representation of the BRST cohomology, are determined by conditions of the form

$$D_i|ph\rangle = 0 \qquad (9.1)$$

where $\{D_i\}$ is a complete set of BRST doublets in involution, or by the conditions

$$D_i^\dagger|ph\rangle = 0 \qquad (9.2)$$

depending on the choice of basis for the original state space from which the projection to the physical states is performed. $D_i^\dagger$ are required to be algebraically independent of D_i and to form together with D_i a complete set of generalized BRST quartets. The conditions (9.1) and (9.2) should always be the consequences of a more invariant formulation like the coBRST one which yields

$$Q|ph\rangle = {}^*Q|ph\rangle = 0 \qquad (9.3)$$

where Q and ${}^*Q$ are the nilpotent BRST and coBRST operators.

From (9.1) and (9.2) we have found that the physical states may be written as follows

$$|ph\rangle = e^{[Q,\psi]}|ph\rangle_0 \qquad (9.4)$$

where ψ is a hermitian fermionic gauge fixing operator, and $|ph\rangle_0$ BRST invariant states determined by a *hermitian* set of BRST doublets in involution. What concerns the unphysical degrees of freedom $|ph\rangle_0$ does not in general belong to an inner product space

although $|ph\rangle$ does. Since the BRST quartets may also be split into two sets of hermitian BRST doublets there are two choices for $|ph\rangle_0$ and the corresponding ψ.

The expression (9.4) might seem to be a way to just rewrite the conditions (9.1) or (9.2) in terms of similar conditions for $|ph\rangle_0$. This is right. However, the point is that the conditions on $|ph\rangle_0$ are much simpler to solve, not the least since $|ph\rangle_0$ is not restricted to be an inner product state. This we have demonstrated by a detailed analysis of general gauge theories given within the BFV formulation. We have analyzed both irreducible and reducible gauge theories of arbitrary rank and found that there always exist one set of solutions, $|ph\rangle_0$, which are trivial BRST invariant states which only depends on the matter variables , and another set, $|ph\rangle_0$, which are solutions of a Dirac quantization. These solutions generalize the solutions for Lie group theories given in [1, 5] but there obtained by means of a bigrading.

There are several aspects of this approach which remains to elaborate. The connections with the coBRST formulation should *e.g.* be further clarified, and the freedom in the choice of gauge fixing fermion ψ should be determined. Notice that we have only given the necessary ingredients of ψ for a given set of $|ph\rangle_0$ solutions. It is clear that ψ may be chosen in many more ways. It remains to give a precise definition of the time evolution in terms of a nontrivial Hamiltonian. Notice that the Hamiltonian has not entered our treatment so far. The generalization to infinite degrees of freedom should also involve some technicalities to be clarified.

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